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MATT-1103 JANUARY 1975

THE FILAMENTATION INSTABILITY IN AN INHOMOGENEOUS PLASMA

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The Filamentation Instability

In An Inhomogeneous Plasma*

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ABSTRACT

Near forward scattering of electromagnetic radiation from ion acoustic waves in the presence of a density gradient is studied. Two absolutely unstable modes are found, neither of which is suppressed by the density gradient.

We begin with the coupled equations describing the modulation instability[1] in an inhomogeneous plasma $\omega_p^2(x) = \omega_p^2(1+x/L)$,

$$\ddot{a}_1 + \nu\dot{a}_1 + \omega_p^2(1 + x/L)a_1 - c^2\nabla^2 a_1 = -\omega_p^2 a_o a_2 \cos(k_o x - \omega_o t)$$

$$\ddot{a}_2 - s^2\nabla^2 a_2 = \mu c^2 \nabla^2 a_1 a_o \cos(k_o x - \omega_o t) \qquad (1)$$

where s is the ion sound speed, $\mu = m/M$, and a_o , a_1 are the amplitudes of the incoming (pump) and scattered electromagnetic waves, respectively, and a_2 the amplitude of the associated ion acoustic wave. Laplace transforming in time with p the Laplace transform variable, $a_2(t) = \int a_2(p)\exp(pt)\,dp$, we find

$$(p^2 + k_\perp^2 s^2)a_2 - s^2(d^2/dx^2)a_2$$
$$= \mu c^2/2\left[(d^2/dx^2) - k_\perp^2\right]\left[a_+ \exp(ik_o x) + a_- \exp(-ik_o x)\right]$$

$$\left[(p \pm i\omega_o)^2 + \nu(p \pm i\omega_o) + c^2 k_\perp^2 + \omega_p^2\right]a_\pm + \omega_p^2(x/L)a_\pm$$
$$- c^2(d^2/dx^2)a_\pm = (-\omega_p^2/2)a_2 a_o \exp(\mp ik_o x) \qquad (2)$$

and $a_\pm = a_1(p \pm i\omega_o)$. We have assumed plane wave structure for the directions perpendicular to the density gradient; $\nabla^2 = -k_\perp^2 + (d^2/dx^2)$. Fourier transforming Eq. (2) with $a_2(k) = \int \exp(-ikx)a_2(x)\,dx$ we find[2]

$$(p^2 + k_\perp^2 s^2)a_2 + k^2 s^2 a_2 = (-\mu/2)c^2(k^2 + k_\perp^2)(a_\pm + a)a_o$$

$$[(p \pm i\omega_o)^2 + \nu(p \pm i\omega_o) + c^2 k_\perp^2 + \omega_p^2 + c^2(k \mp k_o)^2]a_\pm$$

$$+ (i\omega_p^2/L)(d/dk)a_\pm = (-\omega_p^2/2)a_2 a_o \qquad (3)$$

where $a_\pm$ now refer to the Fourier-Laplace components of a_1 evaluated at $p \pm i\omega_o$, $k \mp k_o$. Eliminating a_2 , we then find

$$(-iL/\omega_p^2)[\pm i\omega_o(2p + \nu) + c^2(k_\perp^2 + k^2) \mp 2kk_o c^2]a_\pm + (d/dk)a_\pm$$

$$= (-iL\mu/4)f(k)(a_\pm + a_-) \qquad (4)$$

where $f(k) = \dfrac{(k^2 + k_\perp^2)V_o^2}{p^2 + k^2 s^2 + k_\perp^2 s^2}$, and for simplicity we have assumed ν , $|p| << k_\perp c$. Eliminating a_+ and then writing $[f' = (df/dk)]$

$$a_- = \psi(k) \exp\left\{\int^k [(iLc^2/\omega_p^2)(k^2 + k_\perp^2) - (iL/4)\mu f + f'/2f]dk'\right\}$$

we find

$$d^2\psi/dk^2 + F^2\psi = 0 \qquad (5)$$

with

$$F^2(k) = \Bigg[f''/f - 3/4(f'/f)^2 + (L/\omega_p^2)(f'/f)\omega_o(2p + \nu) - 2iLc^2 k_o/\omega_p^2$$

$$+ \frac{L^2}{\omega_p^4}[i\omega_o(2p + \nu) - 2c^2 kk_o]^2 + \frac{L^2}{16}\mu^2\left(\frac{(k^2 + k_\perp^2)V_o^2}{p^2 + k^2 s^2 + k_\perp^2 s^2}\right)^2\Bigg] . \qquad (6)$$

For large $|k|$ $F^2 \to L^2/\omega_p^4[i\omega_o(2p+\nu) - 2c^2kk_o]^2$ and thus the WKB solutions $\psi_\pm = (1/F^{1/2})\exp(\pm i\int^k Fdk')$ have the asymptotic behavior

$$\psi_\pm \to \frac{1}{\sqrt{k}} \exp\left(\mp \frac{ic^2k_oLk^2}{\omega_p^2} \mp \frac{L\omega_o(2p+\nu)k}{\omega_p^2} \right) . \qquad (7)$$

These solutions are well behaved at $k \to \pm\infty$, respectively, $2\mathrm{Re}p + \nu > 0$. We therefore look for turning points such that these asymptotic solutions can be smoothly joined to yield an eigenvalue solution to Eq. (5). Consider first the case $L \to \infty$. If turning points k_1, k_2 are found such that a well-behaved solution of Eq. (5) can be constructed, the eigenvalue equation is of the form $\int_{k_1}^{k_2} Fdk = (n + 1/2)\pi$. Since for $L \to \infty$ $F \sim L$ this reduces to the simple requirement that the turning points be coincident. We will consider the density gradient as a perturbation on the $L = \infty$ (homogeneous) case. From Eq. (6), the condition that $F^2(k) = 0$ for $L = \infty$ gives

$$[i\omega_o(2p+\nu) - 2c^2k_ok](p^2 + k_\perp^2s^2 + k^2s^2) = i\omega_p^2(k_\perp^2 + k^2)V_o^2\mu/4 . \quad (8)$$

Differentiating Eq. (6) we find $\partial F^2/\partial k = 0$ for $L \to \infty$ implies

$$\frac{-4c^2k_o}{\omega_p^4}[i\omega_o(2p+\nu) - 2c^2k_ok] + \frac{\mu^2}{4}\frac{V_o^4(k^2 + k_\perp^2)kp^2}{(p^2 + k^2s^2 + k_\perp^2s^2)^3} = 0 . \qquad (9)$$

Eliminating V_o from Eqs. (8) and (9), we find the condition that the turning points be coincident to be either

$$-c^2k_o(k_\perp^2 + k^2)(p^2 + k_\perp^2s^2 + k^2s^2) = kp^2[i\omega_o(2p+\nu) - 2c^2kk_o] \qquad (10a)$$

or $$2k_o c^2 k = i\omega_o(2p + \nu) \ . \tag{10b}$$

Considering the first possibility, if $|p| << k_\perp s$ it is clear that there is no solution to Eq. (10a) unless $k \to \pm ik_\perp$, and in fact we find $k = -ik_\perp + p/(\sqrt{2}\ s)$. However, an examination of the Anti-Stokes lines in the vicinity of this point shows that it does not correspond to an absolute instability. No well-behaved solution can be constructed. We thus consider $|p| >> k_\perp s$. Equation (10a) then reduces to a quadratic in k with solution

$$\frac{k}{ik_\perp} = \frac{\omega_o(2p + \nu)}{2k_o k_\perp c^2} \pm \left(\frac{\omega_o^2(2p + \nu)^2}{4k_o^2 k_\perp^2 c^4} - 1\right)^{1/2} \ . \tag{11}$$

Using $|2p + \nu| << k_\perp c$, we then find from Eq. (8)

$$p_\infty = \sqrt{i}(\omega_{pi}/2)(V_o/c)(k_\perp/k_o)^{1/2} \tag{12}$$

where $\omega_{pi} = \omega_p(m/M)^{1/2}$. Since the WKB solution has large components in k-space at $k = \pm k_\perp$, the associated ion acoustic waves are propagating at 45^o to the density gradient.

We now increase the density gradient. As we are interested in large $k_o L$ and $|p| << k_\perp c$, we neglect the f'/f corrections to $F^2(k)$ in Eq. (6) which are easily seen to be corrections of second order in the quantities $(k_o L)^{-1}$, $p/k_\perp c$. Perturbing about the $L = \infty$ case we have for k in the vicinity of $k_\infty \approx \pm k_\perp$

$$F^2 = (\partial^2 F_\infty^2/\partial k^2)(k - k_\infty)^2/2 + (\partial F_\infty^2/\partial p)\Delta p + [\partial F_\infty^2/\partial(1/L)](1/L) \tag{13}$$

where the subscript indicates that F^2 and the derivatives are to be evaluated for $L \to \infty$, and at the value of p , k for which F_∞^2 has coincident zeros.

From Eq. (6), we find

$$\frac{\partial^2 F^2}{\partial k^2} = \frac{8L^2c^4k_o^2}{\omega_p^4} + \frac{L^2\mu^2V_o^4k_\perp^2}{p^4} \tag{14}$$

which reduces, using Eq. (12), to

$\frac{\partial^2 F_\infty^2}{\partial k^2} = \frac{-8L^2c^4k_o^2}{\omega_p^4}$. Similarly find $\frac{\partial F^2}{\partial p} = \frac{16L^2c^4k_o^2k_\perp^2}{\omega_p^4 p}$ and thus

$$F^2 = \frac{-4L^2c^4k_o^2q^2}{\omega_p^4} + \frac{16L^2c^4k_o^2k_\perp^2}{\omega_p^4 p}\Delta p - \frac{2iLc^2k_o}{\omega_p^2} + \frac{2}{k_\perp^2} \tag{15}$$

where $q = k - k_\infty$. Writing $F^2 = c(q_T^2 - q^2)$ the eigenvalue equation $\int F dq = (\pi/2)(2n+1)$ reduces to $c^{1/2}q_T^2 = 2n + 1$ which gives $\frac{\Delta p}{p} = \frac{\omega_p^2(2n + 1 + i)}{8k_\perp^2c^2k_oL}$ and thus, using Eq. (12)

$$\frac{\Delta\gamma}{\gamma} = \frac{n\omega_p^2}{4k_\perp^2c^2k_oL} \quad . \tag{16}$$

The density gradient contributes a frequency shift but no change in the growth rate to the lowest ($n = 0$) mode, and further destabilizes the higher modes. From the value of q_T , we find $\Delta k = \omega_p/[c(k_oL)^{1/2}](n + 1/2)^{1/2}$ and thus $\Delta x/L = k_oc/[(k_oL)^{1/2}\omega p](n + 1/2)^{-1/2}$ for the width of the unstable region. In Fig. 1 are shown the turning points and Anti-Stokes lines for a typical case in a finite density gradient.

It is also possible for roots k_2 amd k_3 of Fig. 1 to become nearly coincident in the vicinity of $k = 0$, leading to a second unstable mode. This near coincidence corresponds to approximate equality of Eq. (10b). From Eq. (6) for $k << k_\perp$ we find (we need not retain the condition $p >> k_\perp s$) ,

$$F^2 \approx \frac{L^2}{\omega_p^4}[i\omega_o(2p+\nu) - 2c^2k_ok]^2 + \frac{L^2\mu^2V_o^4k_\perp^4}{16(p^2+k_\perp^2s^2)^2} - \frac{2iLc^2k_o}{\omega_p^2} + \frac{2}{k_\perp^2} \qquad (17)$$

where we have again neglected the f'/f terms. The eigenvalue for the lowest mode is given by $\int_{k_2}^{k_3} F dk = (\pi/2)(2n+1)$. Write $F^2 = c(q_T^2 - q^2)$ with $q = k - [i\omega_o(2p + \nu)]/(2k_oc^2)$ and find $cq_T^2 = (2n + 1)\sqrt{c}$, or

$$\frac{2}{k_\perp^2} + \frac{L^2\mu^2V_o^4k_\perp^4}{16(p^2 + k_\perp^2s^2)^2} - \frac{2iLc^2k_o}{\omega_p^2} = -\frac{2iLc^2k_o}{\omega_p^2}(2n + 1) \ . \qquad (18)$$

An examination of the Anti-Stokes structure shows that the choice $q_T^2 \sim i$ leads to a solution well behaved at $k \to \pm\infty$, whereas $q_T^2 \sim -i$ does not, subdominant regions not containing the asymptotic real k - axis. Choosing the branch $\sqrt{c} \sim +i$ this means that $n \leq -1$ for a well-behaved solution. We then find

$$(p^2 + k_\perp^2s^2)^2 = \frac{i\omega_p^2V_o^4\mu^2Lk_\perp^4}{64k_oc^2n}\left(1 + \frac{i\omega_p^2}{2k_oLk_\perp^2c^2n}\right)^{-1}$$

and the most rapidly growing mode ($n = -1$) gives ($k_oL >> \omega_p^2/k_\perp^2c^2$, $|p| >> k_\perp s$)

$$p = \frac{\exp(-\pi i/8)\,(k_oL)^{1/4}}{\sqrt{8}} \sqrt{\mu}\,\frac{V_o}{c}\left(\frac{\omega_p}{k_oc}\right)^{1/2} k_\perp c \ . \qquad (19) \ .$$

In this case, we find for the width of the unstable region $\Delta k = \omega_p/[c(k_oL)^{1/2}]$ and $\Delta x/L = k_oc/[\omega_p(k_oL)^{1/2}]$. As we initially assumed $k << k_\perp$, we find the condition $k_o L >> \left(\frac{\omega_p}{k_\perp c}\right)^2$.

The Anti-Stokes lines and turning points for a typical case ($p > k_\perp s$) are shown in Fig. 2. The structure is somewhat complicated due to the pressure of the turning points k_1 , k_4 , but a well-behaved solution can be constructed. Using the notation of Heading,[2] begin in regin A (see Fig. 2) with the solution $\psi(x) = (k,k_4)_s$ which is subdominant on the real axis provided $\mathrm{Re}(2q + \nu) > 0$ in which case the Anti-Stokes line connected to k_4 crosses the real axis as shown. Passing to region B , we find $\psi(k) = (k,k_4)_d$. Moving to C , this solution remains dominant with respect to k_4 regardless of whether the Stokes line emanating from k_4 is crossed or not. We then transfer to k_3 using $(k,k_4)_d = (k,k_3)_3(k_3,k_4)$. The solution thus remains purely subdominant with respect to k_3 . This completes the construction since by a similar continuation from $k = -\infty$ to k_2 we are left with the harmonic oscillator-like problem of connecting the solutions $(k,k_3)_s$, subdominant to the right of k_3 , and $(k_2,k)_s$, subdominant to the left of k_2 . This gives the usual eigenvalue condition $\int_{k_2}^{k_3} F dk = (2n + 1)(\pi/2)$.

ACKNOWLEDGMENTS

The authors wish to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, Italy, where this work was begun.

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*Research supported in part by the U. S. Atomic Energy Commission, Contract AT(11-1)-3237.

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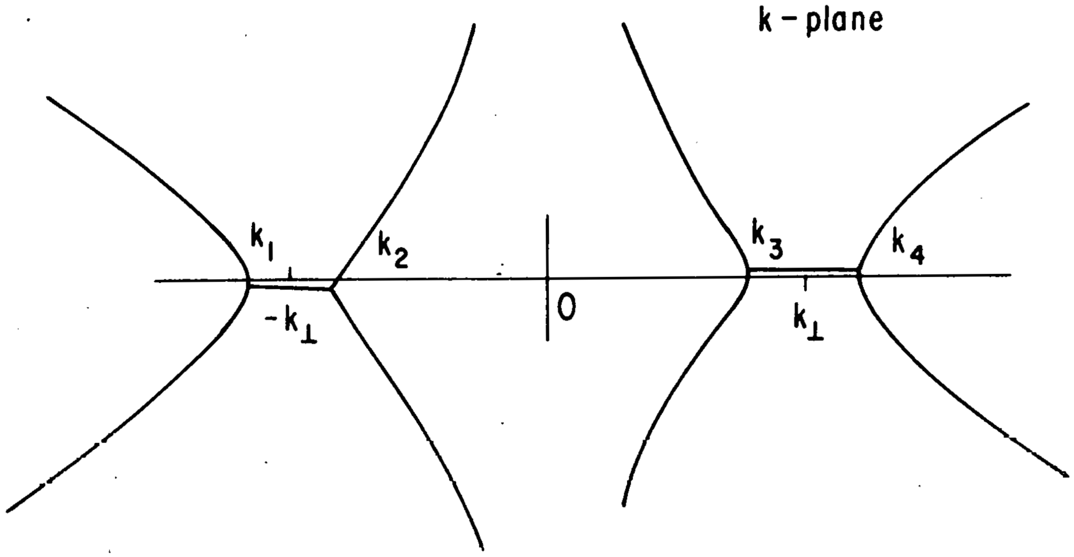

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Fig. 1. Turning points and Anti-Stokes lines for an unstable mode with ion acoustic waves at an angle of 45^o to the incident beam. Parameters used were $s/c = 10^{-4}$, $k_\perp = 0.1\ k_o$, $\omega_p^2/\omega_o^2 = 0.2$, $V_o/c = 0.17$, $k_oL = 500$. The eigenvalue found was $p/\omega_o = (2.12 + 2.14i) \times 10^{-4}$, in agreement with Eq. (12) and Eq. (16).

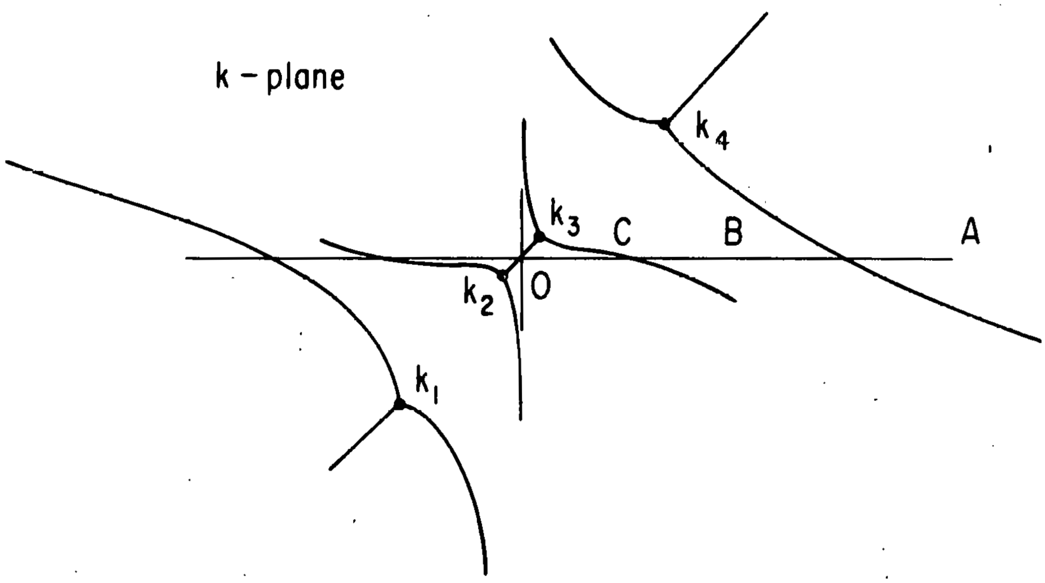

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Fig. 2. Turning points and Anti-Stokes lines for an unstable mode with ion acoustic waves perpendicular to the incident beam. Parameters used were $s/c = 10^{-3}$, $k_\perp/k_o = 10^{-2}$, $\omega_p^2/\omega_o^2 = 0.2$, $V_o/c = 0.52$, $k_oL = 2000$. The eigenvalue found was $p/\omega_o \approx 3 \times 10^{-4} \cdot \exp(-i\pi/8)$ in agreement with Eq. (19).

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